Simplex method

There is a lot of on-line material on simplex method, with solved examples and so on; Oskar Kędzierski's lecture slides are just one example. I strongly recommend looking for it yourself.

You can find a description of the simplex method below. It is illustrated by a problem with $n = 4$ variables and $m = 2$ constraints.

Before you start:

1. Find the standard form of your problem

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 - c \to \min, \text{ under conditions}$$
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \end{cases}$$
$$x_1, x_2, x_3, x_4 \geqslant 0$$

   Usually $c = 0$, as this value does not affect the process.

2. Build a simplex tableau as follows:

$$\left[\begin{array}{cccc|c} c_1 & c_2 & c_3 & c_4 & c \\ \hline a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \end{array}\right]$$

   The bottom rows correspond to the constraints (a system of linear equations), while the top row corresponds to the objective function we are minimizing. This description remains valid during the whole procedure.

3. Find some basic feasible set, in our case e.g. $\mathcal{B} = \{1, 3\}$. By this we mean that the system has a solution of the form $(x_1, 0, x_3)$ with $x_1, x_3 \geqslant 0$.

   You can find such a set just by checking all possibilities. There is also a separate method for that.

Simplex method:

1. By elementary operations on rows, solve the system with respect to the basic variables. For $\mathcal{B} = \{1, 3\}$, this means reaching the following form:

$$\left[\begin{array}{cccc|c} c_1 & c_2 & c_3 & c_4 & c \\ \hline 1 & a_{12} & 0 & a_{14} & b_1 \\ 0 & a_{22} & 1 & a_{24} & b_2 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cccc|c} c_1 & c_2 & c_3 & c_4 & c \\ \hline 0 & a_{12} & 1 & a_{14} & b_1 \\ 1 & a_{22} & 0 & a_{24} & b_2 \end{array}\right]$$

   (with possibly changed a_{ij}, b_i, c_j). It is easy to obtain one form from the other (just by swapping rows), but not necessary.

   Also, reduce c_j's in basic columns (by subtracting from the top row), i.e. reach

$$\left[\begin{array}{cccc|c} 0 & c_2 & 0 & c_4 & c \\ \hline 1 & a_{12} & 0 & a_{14} & b_1 \\ 0 & a_{22} & 1 & a_{24} & b_2 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cccc|c} 0 & c_2 & 0 & c_4 & c \\ \hline 0 & a_{12} & 1 & a_{14} & b_1 \\ 1 & a_{22} & 0 & a_{24} & b_2 \end{array}\right]$$

2. If $c_j \geqslant 0$ for all $j = 1, \ldots, 4$, the minimum is $-c$ and our basic solution is a minimizer (we just need to read it from the tableau). In the examples above, it is respectively $(b_1, 0, b_2, 0)$ or $(b_2, 0, b_1, 0)$. STOP!

3. If not, choose the smallest of the coefficients c_j, let it be $c_s < 0$.

4. If $a_{is} \leqslant 0$ for all $i = 1, 2$ (i.e., all coefficients below c_s are nonpositive), our objective function is unbounded from below and there is no minimum. STOP!

5. If not, consider all i for which $a_{is} > 0$ and choose $r \in \{1, 2\}$ such that $\frac{b_r}{a_{rs}}$ is minimal among all such ratios (we do not consider ratios with $a_{is} \leqslant 0$ at all).

6. To form the new basic set $\mathcal{B}'$, add s as a new basic column, and remove the basic column which has 1 in r-th row. For example, if the tableau has the form

$$\left[\begin{array}{cccc|c} 0 & c_2 & 0 & c_4 & c \\ \hline 0 & a_{12} & 1 & a_{14} & b_1 \\ 1 & a_{22} & 0 & a_{24} & b_2 \end{array}\right],$$

   $s = 4$ and $r = 2$, the new basic set is $\mathcal{B}' = \{3, 4\}$. We add 4 (because $c_4 < 0$ was the smallest) and remove 1 (because $\frac{b_2}{a_{24}}$ was the smallest and 1 appears in the first column, not the third).

7. Start from the beginning with $\mathcal{B}'$ instead of $\mathcal{B}$. This is not so bad, as the system is almost solved – in the example, we just need to get $0, 0, 1$ in the fourth column.